NOVEMBER/DECEMBER 2019

MMA31 — COMPLEX ANALYSIS — I

Time : Three hours Maximum : 75 marks

SECTION A — (5 × 6 = 30 marks)

Answer ALL questions.

1. (a) Compute $\int_{\gamma} x\,dz$, where γ is the directed line segment from 0 to $1+i$.

Or

(b) Compute $\int_{|z|=1} |z-1| \cdot |dz|$.

2. (a) Evaluate $\int_{|z|=\rho} \frac{dz}{|z-a|^2}, |a| \neq \rho$.

Or

(b) State and prove Cauchy's integral formula.

3. (a) If $f(z)$ is analytic in Ω, then prove that $\int_{\gamma} f(z)\,dz = 0$ for every cycle γ which is homologous to zero in Ω.

Or

(b) State and prove Argument theorem.

4. (a) If u_1 and u_2 are harmonic in a region Ω, then prove that $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$.

Or

(b) Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ using residue theorem.

5. (a) State and prove Schwarz's theorem.

Or

(b) Using Taylor's theorem applied to a branch of $\log(1 + z/n)$. Prove that $\lim_{n\to\infty} (1 + z/n)^n = e^z$.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. Prove that the line integral $\int_\gamma p\,dx + q\,dy$, defined in Ω, depends only on the end points of γ if and only if there exist a function $U(x, y)$ in Ω with the partial derivatives $\frac{\partial U}{\partial x} = p$, $\frac{\partial U}{\partial y} = q$.

7. If the piecewise differentiable closed curve γ does not pass through the point a, then prove that the value of the integral $\int_\gamma \frac{dz}{z - a}$ is a multiple of $2\pi i$.

8. State and prove the residue theorem.

9. Suppose that $u(z)$ is harmonic for $|Z| < R$, continuous for $|Z| \leq R$. Prove that $u(a) = \frac{1}{2\pi} \int_{|Z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta$ for all $|a| < R$.

10. If $f(z)$ is analytic in the region Ω containing z_0 then prove that the representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f'' z_0}{2!}(z - z_0)^2 + \ldots + (z - z_0)^n \frac{f^n(z_0)}{n!} + \cdots$$

is valid in largest disc of centre z_0 obtained in Ω.