9. Assume that $\sum_{n=0}^{\infty} a_n$ converges absolutely and has sum A and suppose $\sum_{n=0}^{\infty} b_n$ converges with sum B. Prove that the Cauchy product of these two series converges and has sum AB.

10. Let α be of bounded variation on $[a,b]$. Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in R(\alpha)$ on $[a,b]$ for each $n = 1,2,\ldots$ Assume that $f_n \to f$ uniformly on $[a,b]$ and define $g_n(x) = \int_a^x f_n(t)d\alpha(t)$ if $x \in [a,b]$, $n = 1,2,\ldots$ prove that the following

(a) $f \in R(\alpha)$ on $[a,b]$

(b) $g_n \to g$ uniformly on $[a,b]$, where $g(x) = \int_a^x f(t)d\alpha(t)$.

APRIL/MAY 2018

MMA12 — REAL ANALYSIS — I

Time : Three hours Maximum : 75 marks

SECTION A — $(5 \times 6 = 30$ marks)

Answer ALL questions.

1. (a) Let f be defined on $[a,b]$. Prove that f is of bounded variation on $[a,b]$ if, and only if, f can be expressed as the difference of two increasing functions.

Or

(b) If f is continuous on $[a,b]$ and if f' exists and is bounded in the interior, say $|f'(x)| \le A$ for all x in (a,b), prove that f is of bounded variation on $[a,b]$.

2. (a) Assume that $\alpha \nearrow$ on $[a,b]$, prove that if p' is finer than p, $\cup(p',f,\alpha) \le \cup(p,f,\alpha)$ and $L(p',f,\alpha) \ge L(p,f,\alpha)$.

Or

(b) If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$, prove that $f \in R(C_1\alpha + C_2\beta)$ on $[a, b]$ and $\int_a^b fd(C_1\alpha + C_2\beta) = C_1\int_a^b fd\alpha + C_2\int_a^b fd\beta$.

3. (a) If f is continuous on $[a, b]$ and if α is of bounded variation on $[a, b]$, prove that $f \in R(\alpha)$ on $[a, b]$.

Or

(b) State and prove first mean–value theorem for Riemann-Stieljes integrals.

4. (a) State and prove Dirichlet's test.

Or

(b) If a series is convergent with sum S, prove that it is also $(C,1)$ summable with Cesaro sum S.

5. (a) Assume that $f_n \to f$ uniformly on S. If each on S. If each f_n is continuous at a point C of S, prove that the limit function f is also continuous at C.

Or

(b) Assume that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ on $[a, b]$. Define $h(x) = \int_a^x f(t)g(t)dt$, $h_n(x) = \int_a^x f_n(t)g_n(t)dt$ if $x \in [a, b]$, prove that $h_n \to h$ uniformly on $[a, b]$.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. Let f be of bounded variation on $[a, b]$. If $x \in (a, b]$, let $v(x) = v_f(a, x)$ and put $V(a) = 0$. Prove that every point of continuity of f is also a point of continuity of v and the converse is also true.

7. Assume that $C \in (a, b)$. If two of the three in the following integrals exists, prove that the third also exists $\int_a^c fd\alpha + \int_c^b fd\alpha = \int_a^b fd\alpha$.

8. Assume that α is of bounded variation on $[a, b]$. Let $v(x)$ denote the total variation of α on $[a, x]$ if $a < x \le b$ and let $v(a) = 0$. Let f be defined and bounded on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$, prove that $f \in R(v)$ on $[a, b]$.