NOVEMBER/DECEMBER 2018

MMA15C — GRAPH THEORY

Time : Three hours Maximum : 75 marks

SECTION A — (5 × 6 = 30 marks)

Answer ALL questions.

1. (a) If G is a tree, prove that $e = v - 1$.

Or

(b) Show that every connected graph contains a spanning tree.

2. (a) If G is a block with $v \geq 3$, prove that any two edges of G lie on a common cycle.

Or

(b) Let G be a simple graph and let u and v be non-adjacent vertices in G such that $d(u) + d(v) \geq v$. Prove that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

3. (a) If G is bipartite, prove that $\chi' = \Delta$.

Or

(b) Let $\mathscr{C} = (E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, prove that the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

4. (a) Prove that $r(k, k) \geq 2^{k/2}$.

Or

(b) Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Prove that $d(u) + d(v) \geq 3k - 5$.

5. (a) If G is a plane graph, prove that $\sum_{f \in F} d(f) = 2e$.

Or

(b) Prove that $K_{3,3}$ is nonplanar.

SECTION B — ($3 \times 15 = 45$ marks)

Answer any THREE questions.

6. Prove that an edge e of G is a cut edge of G if and only if e is contained in no cycle of G.

7. If G is a simple graph with $\upsilon \geq 3$ and $\delta \geq \upsilon/2$, prove that G is Hamiltonian.

8. Let G be a bipartite graph with bipartition (X, Y). Prove that G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

9. Let G be a k-critical graph with 2-vertex cut $\{u, v\}$. Prove that

(a) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of type i $(i = 1, 2)$.

(b) Both $G_1 + uv$ and $G_2 \cdot uv$ are k-critical.

10. Prove that the following :

(a) Every planar graph is 4-vertex-colourable.

(b) Every plane graph is 4-face-colourable.

(c) Every simple 2-edge-connected 3-regular planar graph is 3-edge-colourable.