8. Prove that a finite Cartesian product of connected spaces is connected.

9. Let X be a nonempty compact Hausdorff space. If X has no isolated points, then prove that X is uncountable.

10. State and prove Urysohn lemma.

NOVEMBER/DECEMBER 2019

MMA32 — TOPOLOGY

Time : Three hours Maximum : 75 marks

SECTION A — ($5 \times 6 = 30$ marks)

Answer ALL questions.

1. (a) If X be a set and $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X, prove that $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Or

(b) If A is a subspace of X and B is a subspace of Y, prove that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

2. (a) State and prove Pasting Lemma.

Or

(b) Let $f : X \to Y$; let X and Y be metrizable with metrics d_x and d_y respectively, then prove that the continuity of f is equivalent to the requirement that given $x \in X$ and given $\in > 0$, there exists $\delta > 0$ such that $d_x(x, y) \Rightarrow d_y(f(x), f(y)) < \in$.

3. (a) Prove that the union of a collection of connected subspaces of X that have a point in common is connected.

Or

(b) Prove that the components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

4. (a) Prove that every compact subspace of a Hausdorff space is closed.

Or

(b) Let $f: X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then prove that f is a homeomorphism.

5. (a) Prove that a subspace of a regular space is regular; a product of regular spaces is regular.

Or

(b) Let X be a topological space. Let one-point sets in X be closed. Prove that X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. Let A be a subset of the topological space X. Then prove that

(a) $x \in \overline{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

7. Let X and Y be topological spaces; let $f: X \to Y$. Then prove that the following are equivalent:

(a) f is continuous

(b) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(c) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(d) For each $x \in X$ and neighbourhood V of $f(X)$ there is a neighbourhood U of x such that $f(U) \subset V$.