(c) The Fourier series can be integrated term by term. That is, for all x we have

$$\int_0^x f(t)\,dt = \frac{a_0 x}{2} + \sum_{n=1}^{+\infty} \int_0^x (a_n \cos nt + b_n \sin nt)\,dt$$

the integrated series being uniformly convergent on every interval, even if the Fourier series in ⊗ diverges.

(d) If the Fourier series in ⊗ converges for some x then it converges to $f(x)$

7. (a) State and prove chain rule (7)

(b) Derive Taylor's formula. (8)

8. State and prove implicit function theorem.

9. (a) If $E_1, E_2,,,,$ are measurable subsets of $[a, b]$ and if $E_1 \subset E_2 \subset E_3 ....$ prove that $\bigcup_{n=1}^{\infty} E_n$ is measurable and $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} E_n$.

(b) Let f be a bounded function on $[a, b]$. Prove that $f \in \mathcal{L}\,[a, b]$ if and only if for each $\in > 0$ there exists a measurable partition P of $[a, b]$ such that $U[f; P] < L[f; P] + \in$

10. Prove that metric space $\mathcal{L}^2\,[a, b]$ is complete.

APRIL/MAY 2019

MMA22 — REAL ANALYSIS — II

Time : Three hours Maximum : 75 marks

SECTION A — (5 × 6 = 30 marks)

Answer ALL questions.

1. (a) Assume that $g(0+)$ exists and suppose that for some $\delta > 0$ the Lebesgue integral $\int_0^{\delta} \frac{g(t) - g(0+)}{t}\,dt$ exists. Prove that

$$\lim_{\alpha\to+\infty} \frac{2}{\pi} \int_0^{\delta} g(t) \frac{\sin \alpha t}{t}\,dt = g(0+).$$

Or

(b) Assume that $\{\phi_0, \phi_1, ...\}$ is orthonormal on I. Let $\{c_n\}$ be any sequence of complex numbers such that $\sum |c_k|^2$ converges. Prove that there is a function f in $L^2(\mathcal{L}\,)$ such that

(i) $(f, \phi_k) = c_k$ for each $k \geq 0$

(ii) $\| f \|^2 = \sum_{k=0}^{\infty} |c_k|^2$.

2. (a) Let S be an open connected subset on R^n and let $f: S \to R^n$ be differentiable at each point of S. If $f'(c) = 0$ for each c in S prove that f is constant on S.

Or

(b) If both partial derivatives $D_r f$ and $D_k f$ exist in an n-ball $B(c;\delta)$ and if both are differentiable at c prove that $D_{r,k} f(c) = D_{k,r} f(c)$.

3. (a) Find and classify the extreme values of the function

$f(x, y) = y^2 + x^2 y + x^4$.

Or

(b) Assume that $f = (f_1, ..., f_n)$ has continuous partial derivatives $D_j f_i$ on an open set in R^n and that the Jacobian determinant $J_f(a) \neq 0$ for some point a in S. Then prove that there is an n – ball $B(a)$ on which f is one-to-one.

4. (a) Prove that the subset E of $[a, b]$ is measurable if and only if $\in > 0$ there exist open sets G_1 and G_2 such that $G_1 \supset E$, $G_2 \supset E$ and $|G_1 \cap G_2| < \in$.

Or

(b) Prove that every bounded measurable function on $[a, b]$ is Lebesgue integral.

5. (a) If f is a bounded function in $\mathcal{L}[a, b]$ and if $a < c < b$ then prove that $f \in \mathcal{L}[a, c]$ and $f \in \mathcal{L}[c, b]$.

Or

(b) State and prove Lebesgue dominated convergence theorem.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. Let f be continuous on $[0, 2\pi]$ and periodic with period 2π. Let $\{s_n\}$ denote the sequence of partial sums of the Fourier series generated by $f(say)$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ ———— } \otimes$$

Then prove that

(a) $\lim_{n \to \infty} s_n = f$ on $[0, 2\pi]$

(b) $\frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$