Prove that the following :

(a) $0 < f(n+1) \le d_{n+1} \le d_n \le f(1)$, for $n = 1, 2, \ldots$

(b) $\lim_{n\to\infty} d_n$ exists

(c) $\sum_{n=1}^{\infty} f(n)$ converges if, and only if, the sequence $\{t_n\}$ converges

(d) $0 \le d_k - \lim_{n\to\infty} d_n \le f(k)$, for $k = 1, 2, \ldots$.

10. State and prove Dirichlet's test for uniform convergence.

NOVEMBER/DECEMBER 2018

MMA12 — REAL ANALYSIS – I

Time : Three hours Maximum : 75 marks

SECTION A — ($5 \times 6 = 30$ marks)

Answer ALL questions.

1. (a) If f is bounded on $[a, b]$, prove that f is of bounded variation on $[a, b]$.

Or

(b) State and prove additive property of total variation.

2. (a) Assume that $\alpha \nearrow$ on $[a, b]$, prove that for any two partitions P_1 and P_2, $L(P, f, \alpha) \le U(P_2, f, \alpha)$.

Or

(b) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$, prove that $c_1 f + c_2 g \in R(\alpha)$ on $[a, b]$.

3. (a) State and prove second mean-value theorem for Riemann-Stieltjes integrals.

Or

(b) Let f be continuous on the rectangle $[a,b]\times[c,d]$. If $g\in R$ on $n\in R$ on $[c,d]$, prove that $\int_a^b\left[\int_c^d g(x)h(y)f(x,y)\,dy\right]dx =$

$$\int_c^d\left[\int_a^b g(x)h(y)f(x,y)\,dx\right]dy$$

4. (a) State and prove Abel's test.

Or

(b) Assume that each $a_n > 0$, prove that the product $\pi(1+a_n)$ converges if, and only if, the series $\sum a_n$ converges.

5. (a) Assume that $\lim_{n\to\infty} f_n = f$ on $[a,b]$. If $g\in R$ on $[a,b]$, define $h(x)=\int_a^x f(t)\,g(t)\,dt$, $h_n(x)=\int_a^x f_n(t)\,g(t)\,dt$, if $x\in[a,b]$, prove that $h_n \to h$ uniformly on $[a,b]$.

Or

(b) Assume that $\sum f_n(x) = f(x)$ (uniformly on S). If each f_n is continuous at a point x_0 of S, prove that f is also continuous at x_0.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. Let f be of bounded variation on $[a,b]$. Let V be defined on $[a,b]$ as follow :

$V(x) = V_f(a,x)$ if $a < x \le b$, $V(a) = a$ prove that

(a) V is an increasing function on $[a,b]$.

(b) $V-f$ is an increasing function on $[a,b]$.

7. If $f\in R(\alpha)$ on $[a,b]$, prove that $\alpha\in R(f)$ on $[a,b]$ and

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df\,(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

8. State and prove the second fundamental theorem of integral calculus.

9. Let f be a positive decreasing function defined on $[1,+\infty)$ such that $\lim_{x\to+\alpha} f(x)=0$ for $n=1,2,\ldots$

define $S_n = \sum_{k=1}^{n} f(k)$, $t_n = \int_1^n f(x)\,dx$, $d_n = s_n - t_n$.