9. (a) If E_1 and E_2 are subsets of $[a, b]$ then prove that

$$\overline{m}E_1 + \overline{m}E_2 \geq \overline{m}(E_1 \cup E_2) + \overline{m}(E_1 \cap E_2)$$

$$\underline{m}E_1 + \underline{m}E_2 \leq \underline{m}(E_1 \cup E_2) + \underline{m}(E_1 \cap E_2)$$

(b) Prove that every bounded measurable function on $[a, b]$ is Lebesgue integrable

10. Prove that the metric space $\mathcal{L}^2\,[a, b]$ is complete.

NOVEMBER/DECEMBER 2018

MMA22 — REAL ANALYSIS - II

Time : Three hours Maximum : 75 marks

SECTION A — (5 × 6 = 30 marks)

Answer ALL questions.

1. (a) Assume $\{\phi_0, \phi_1, ...\}$ is orthonormal on I. Let $\{c_n\}$ be any sequence of complex numbers such that $\sum |c_k|^2$ converges. Then prove that there is a function f in $L^2(I)$ such that

(i) $(f, \phi_k) = c_k$ for each $k \geq 0$

(ii) $\|f\|^2 = \sum_{k=0}^{\infty} |c_k|^2$.

Or

(b) Assume that $f \in L(I)$. The prove that

$\lim_{\alpha \to +\infty} \int_I f(t) \sin(\alpha t + \beta)\, dt = 0$, for each real β.

2. (a) Let u and v be two real-valued functions defined on a subset S of the complex plane. Assume that u and v are differentiable at an interior poin c of S and that the partial derivatives satisfy the Cauchy-Riemann equations at c. Then prove that $f = u + iv$ has derivative at c.

Or

(b) State and prove Mean-value theorem.

3. (a) Let A be an open subset of R^n and assume that $f : A \to R^n$ has continuous partial derivatives $D_j f_i$ on A. If $J_f(x) \neq 0$ for all x in A then prove that f is an open mapping.

Or

(b) A quadratic surface with center at the origin has the equation $Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2fxy = 1$. Find the length of its semi-axes.

4. (a) If $E \subset [a,b]$ then prove that $\overline{m}E + \underline{m}E' = b - a$ where $E' = [a,b] - E$.

Or

(b) If f and g are measurable functions on [a, b] then prove that $f + g$ and fg are measurable functions.

5. (a) Let $f \in \mathcal{L}[a,b]$. Then prove that given $\in > 0$, there exist $\delta > 0$ such that $\left|\int_E f\right| < E$ whenever E is a measurable subset of [a, b] with $mE < \delta$.

Or

(b) State and prove Fatou's lemma

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. (a) State and prove the theorem on Best Approximation.

(b) Let f be real-valued and continuous on a compact interval $[a, b]$. Then prove that for every $\in > 0$ there is a polynomial p such that $|f(x) - p(x)| < \in$ for every x in $[a, b]$

7. (a) State and prove the chain rule.

(b) If both partial derivatives $D_r f$ and $D_k f$ exist in an n-ball $B(c;\delta)$ and if both are differentiable at c then prove that $D_{r,k} f(c) = D_{k,r} f(c)$.

8. State and prove the inverse function theorem