APRIL/MAY 2019

MMA21 — ALGEBRA II

Time : Three hours

Maximum : 75 marks

SECTION A — ($5 \times 6 = 30$ marks)

Answer ALL questions.

1. (a) Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

Or

(b) If $a \in K$ is a algebraic of degree n over F prove that $[F(a):F]=n$.

2. (a) Prove that a polynomial of degree n over a field can have atmost n roots in any extension field.

Or

(b) If $p(x)$ is irreducible in $F[x]$ and if v is a root of $p(x)$ prove that $F(v)$ is isomorphic to $F'(\omega)$ where ω is a root of $p'(t)$.

3. (a) If K is finite extensions of F prove that $G(K,F)$ is a finite group and its order $o(G(K,F))$ satisfies $o(G(K,F)) \le [K:F]$.

Or

(b) Let K be the splitting field of $f(x)$ in $F[x]$ and let $p(x)$ be an irreducible factor of $f(x)$ in $F[x]$. If the roots of $p(x)$ are $\alpha_1, \alpha_2 \ldots \alpha_\gamma$ prove that for each i there exists an automorphism σ_i in $G(K,F)$ such that $\sigma_i(\alpha_1) = \alpha_i$.

4. (a) Suppose that the field F has all nth roots of unity and suppose that $a \neq 0$ is in F. Let $x^n - a \in F[x]$ and let K be its splitting over F. Prove that

(i) $K = F(u)$ where u is any root off $x^n - a$

(ii) The Galois group off $x^n - a$ over F is abelian.

Or

(b) State and prove Jacobson theorem.

5. (a) Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. Prove that $D = C$.

Or

(b) Prove that every positive integer can be expressed as the sum of squares of four integers.

SECTION B — ($3 \times 15 = 45$ marks)

Answer any THREE questions.

6. (a) If L is a finite extension of K and if K is a finite extension of F prove that L is a finite extension of F

(b) If a and b in K are algebraic over F of degree m and n respectively prove that $a \pm b, ab$ and a/b are algebraic over F of degree at most m,n.

7. If F is of characteristic 0 and if a,b are algebraic over F prove there exists an element $c \in F(a,b)$ such that $F(a,b) = F(c)$.

8. State and prove the fundamental theorem of Galois theory.

9. Prove that a finite division ring is necessarily a commutative field.

10. State and prove Frobenius theorem.