NOVEMBER/DECEMBER 2018

MMA11 — ALGEBRA — I

Time : Three hours | Maximum : 75 marks

SECTION A — $(5 \times 6 = 30$ marks)

Answer ALL questions.

1. (a) Prove that $a \in z$ if and only if $N(a) = G$. If G is finite, $a \in z$ if and only if $o(N(a)) = o(G)$.

Or

(b) If $o(G) = p^2$, where p is prime number, prove that G is abelian.

2. (a) Let G be a group and suppose that G is the internal direct product of $N_1, N_2, \ldots N_n$. Let $T = N_1 \times N_2 \times \ldots \times N_n$. Prove that G and T are isomorphic.

Or

(b) Suppose that G is the internal direct product of $N_1, N_2, \ldots, N_n$. Prove that for $i \neq j$, $N_i \cap N_j + (e)$ and if $a \in N_i$, $b \in N_j$ and $ab = ba$.

3. (a) Prove that the two nilpotent linear transformations are similar if and only if they have the same invariants.

Or

(b) If $T \in A(V)$ is nilpotent, prove that $\alpha_0 + \alpha_1 T + ... + \alpha_m T^m$, where the $\alpha_i \in F$, is invertible if $\alpha_0 \neq 0$.

4. (a) Prove that for each $i = 1, 2, ...k$, $V_i \neq (0)$ and $V = V_1 \oplus V_2 \oplus .... \oplus V_k$. The minimal polynomial of T_i is $q_i(x)^{li}$.

Or

(b) Suppose the two matrices A, B in F are similar in K_n, where K is an extension of F. Prove that A and B are similar in F_n

5. (a) If A is invertible, prove that $(ACA^{-1}) = trC$.

Or

(b) If $(vT, vT) = (v, v)$ for all $v \in V$, prove that T is unitary.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. State and prove Sylow's theorem.

7. Let G be a group and suppose that G is the internal direct product of $N_1, ...N_n$. Let $T = N_1 \times N_2 \times .... \times N_n$. Prove that G and T are isomorphic.

8. If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular.

9. Let V and W be two vector spaces over F and suppose that ψ is a vector space isomorphism of V onto W. Suppose that $S \in A_F(V)$ and $T \in A_F(W)$ are such that for any $v \in V$, $(vS)\psi = (v\psi)T$. Prove that S and T have the same elementary divisors.

10. Prove that $T \geq 0$ if and only if $T = AA^*$ for some A.
