Time : Three hours   Maximum : 75 marks

SECTION A — (5 × 6 = 30 marks)

Answer ALL questions.

1. (a) Prove that the infinite product $\prod_{1}(1+\alpha_n)$ with $1+\alpha_n \neq 0$ converges simultaneously with the series $\sum_{1}^{\infty} \log(1+\alpha_n)$ whose terms represent the values of the principal branch of the logarithm.

Or

   (b) Show that $\prod_{n=2}^{\infty}\left(1-\dfrac{1}{n^2}\right)=\dfrac{1}{2}$.

2. (a) Prove that for $\sigma = \operatorname{Re} s > 1$,
$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}(1-p_n^{-s}).$$

Or

414

(b) Prove that a family F is normal if and only if its closure F^- with respect to the distance function $\rho(f, g) = \sum_{k=1}^{\infty} \delta_k(f, g) 2^{-k}$ is compact.

3. (a) Prove that a continuous function $u(z)$ which satisfies condition

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta$$

is necessarily harmonic.

Or

(b) Describe the Harnack's principle.

4. (a) Show that an elliptic function without poles is constant.

Or

(b) Prove that the zeros $a_1, \ldots, a_n$ and poles $b_1, \ldots, b_n$ of an elliptic function satisfy $a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{M}$.

5. (a) Prove that the modular function $\lambda(r)$ effects a one-to-one conformal mapping of the region Ω onto the upper half plane. The mapping extends continuously to the boundary in such a way that $\tau = 0, 1, \infty$ correspond to $\lambda = 1, \infty, 0$.

Or

2

414

(b) Show that any elliptic function with periods ω_1, ω_2 can be written as $C \prod_{k=1}^{n} \dfrac{\sigma(z - a_k)}{\sigma(z - b_k)}$ $(C = \text{const.})$

SECTION B — $(3 \times 15 = 45$ marks$)$

Answer any THREE questions.

6. Derive Jensen's formula.

7. Prove that $\zeta(s) = 2^s \pi^{s-1} \sin\dfrac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$.

8. Prove that given any simply connected region Ω which is not the whole plane, and a point $z_0 \in \Omega$, there exists a unique analytic function $f(z)$ in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that $f(z)$ defines a one-to-one mapping of Ω onto the disk $|w| < 1$.

9. Prove that there exists a basis (ω_1, ω_2) such that the ratio $\tau = \omega_2 / \omega_1$ satisfies the following conditions : (a) $\text{Im}\,\tau > 0$, (b) $-\dfrac{1}{2} < \text{Re}\,\tau \le \dfrac{1}{2}$, (c) $|\tau| \ge 1$, (d) $\text{Re}\,\tau \ge 0$ if $|\tau| = 1$. The ratio τ is uniquely determined by these conditions, and there is a choice of two, four, or six corresponding bases.

10. Derive the Legendre's relation.

_____

3

414