8. Prove that If $P_1, P_2, ...P_n$ are the projections on closed linear sub spaces $M_1, M_2, ...M_n$ of H, then $P = P_1 + P_2 + ... + P_n$ is a projection $\Leftrightarrow$ the P is are pairwise orthogonal (in the sense that $P_i P_j = 0$ whenever $i \neq j$); and in this case P is the projection on

$$M = M_1 + M_2 + ... + M_n.$$

9. Prove that If r is an element of A with the property that $1 - xr$ is regular for every x, then r is in R.

10. Prove that If A is a commutative B*-algebra, then the Gelfand mapping $x \to \hat{x}$ is an isometric *-isomorphism of A onto the commutative B*-algebra in $\mathfrak{C}(\mathfrak{M})$.

NOVEMBER/DECEMBER 2019

MMA42 — FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 75 marks

SECTION A — (5 × 6 = 30 marks)

Answer ALL questions.

1. (a) Prove that If N and N' be a normed linear spaces, and T a linear transformation of N into N'. Then the following condition on T are all equivalent to one another.

   (i) T is continuous

   (ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_0) \to 0$;

   Or

   (b) Prove that in a normed linear space, addition and scalar multiplication are jointly continuous.

2. (a) Prove that Let M be a closed linear subspace of a Hibert space H, let x be a vector not in M, and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

Or

(b) Prove that if M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$.

3. (a) Prove that If P is the projection on a closed linear subspace M of H, the M reduces an operator $T \Leftrightarrow TP = PT$.

Or

(b) Prove that If N_1 and N_2, are normal operators on H with the property that either commutes with the adjoint of the other, then $N_1 + N_2$, and $N_1 N_2$, are normal.

4. (a) If r is an element of R, prove that 1— r is left regular.

Or

(b) Prove that A/R is a semi — simple Banach algebra.

5. (a) Prove that If x is a normal element in a B*-algebra, then $\|x^2\| = \|x\|^2$.

Or

(b) Prove that If A is self -adjoint, and if $\|x^2\| = \|x\|^2$ for every x, then the Gelfand mapping $x \to \hat{x}$ is an isometric isomorphism of A onto $\mathcal{C}(\mathfrak{M})$.

SECTION B — (3 × 15 = 45 marks)

Answer any THREE questions.

6. Prove that Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if

$$M_0 = M + [x_0]$$

is the linear subspace spanned by M and x_0, then f can be extended to a function f_0, defined on M_0 such that $\|f_0\| = \|f\|$.

7. Prove that A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.